South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 2 (2025), pp. 253-270

DOI: 10.56827/SEAJMMS.2025.2102.18 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

TRIPLED FIXED POINTS IN COMPLETE E-CONE METRIC SPACES

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(**Received:** *Oct. 03, 2024* **Accepted:** *Jul. 26, 2025* **Published:** *Aug. 30, 2025)*

Abstract: In this paper, we investigate the tripled fixed point results at in the context of complete $E-$cone metric spaces. We extend existing fixed point theorems by establishing conditions under which a mapping possesses a unique tripled fixed point. Utilizing novel contractive conditions, we develop new criteria that guarantee the existence and uniqueness of such fixed points. These results broaden the scope of fixed-point theory and offer new tools for addressing problems in applied mathematics, particularly in the areas of nonlinear analysis, dynamic systems, and optimization. Our results help us learn more about fixed points in more general structures. This makes it possible for more research to be done in generalized metric spaces and how they can be used to solve differential equations, integral equations, and boundary value problems.

Keywords and Phrases: Tripled fixed points, $E-$cone metric spaces, Contractive conditions.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

An intriguing extension of the classical metric space idea is partially E-cone spaces. The origin of E-cone metric spaces is defining the metric using a particular cone in Banach space [15]. This structure provides non-negative scalar that supports the determination of distance in a reacher context more effectively than using a real number alone [4, 6, 19, 31, 33, 34, 35, 37, 38].

In the study of metric geometry, partially E-cone metric spaces(see [18]) which combine the partial ordering and the element of cone metric spaces represent an exciting advance. Introduced as a generalization of metric spaces in which distance is measured inside a cone in the Banach space [15] rather than in the standard set of non-negative real numbers, these spaces provide an additional mathematical extension. The key feature of partially E-cone metric spaces is their capacity to integrate an order relation into the structure of cone metric spaces. Compared to conventional metric spaces, this novel method offers more versatility and wider applicability.

Metric spaces are a useful idea in topology and mathematical analysis because they provide a structural framework for defining continuity and distance. This opens up new possibilities, especially in fields where order relationships are important, such theoretical economics and fixed point theory. One might consult the work of Kadalburg, Radenovic, and Rakocevic, who offer thorough studies in this field, especially in their paper "Revisiting cone metric spaces and fixed point theorem of contractive mapping" [5, 20, 21, 25] and the references therein, for a foundational discussion on E-cone metric spaces.

Recently, there has been additional investigation on partially E-cone metric spaces with the aim of strengthening their theoretical basis and expanding their practical scope. These spaces are an advanced version of metric spaces that establish distance by using a partial ordering in conjunction with a cone inside of a Banach space [15]. These frameworks integrate ordering structures, which are essential for mathematical models including optimization and hierarchy sensitive processes, in addition to enriching the traditional concepts of distance and conversion. The work of Aydi and Karapinar [9], whose authors created fixed point theorems that are fundamental to mathematical analysis and algorithms in such structured space, provides a current thorough assessment and advancement in this topic.

Applications of the Banach Fixed point theorem are widely recognized. Numerous scholars have expanded upon this theorem by incorporating broader contractive conditions that necessitate the presence of a fixed point. Nieto and Lopez [30] later studied the existence of fixed points in ordered metric spaces, after Ran and Reurings [32] in 2004. Regarding additional outcomes in ordered metric spaces, see e.g. [3, 8, 13, 26, 27-29]. Bhashkar and Lakshmikantham [17] explored several coupled Fixed point theorems in partially ordered complete metric spaces and established the idea of a coupled Fixed point of a mapping $F : X \times X \to X$. Subsequently, different outcomes regarding coupled fixed points were discovered, see e.g. [2, 10, 13, 14, 22, 23, 24, 29, 30]. On the other hand, tripled fixed point theory was first presented by Berinde and Borcut. [16] (see also the papers [1, 11]).

In this paper we define the concept of triple fixed point on E-cone metric space.

2. Preliminaries

Definition 2.1. [7] *Let $\xi \neq \phi$ and consider an ordered space E over the real scalars. An E-valued function $d^E : \xi \times \xi \to E$ such that for all $\hbar, \vartheta, \varrho \in \xi$, then*

(i) $0_E \leq d^E(\hbar, \vartheta), d^E(\hbar, \vartheta) = 0_E$ *if and only if* $\hbar = \vartheta$;

(ii) $d^E(\hbar, \vartheta) = d^E(\vartheta, \hbar)$;

(iii) $d^E(\hbar, \vartheta) \leq d^E(\hbar, \varrho) + d^E(\varrho, \vartheta)$.

Then the pair (ξ, d^E) is called E-metric space.

Definition 2.2. [18] *Let $\xi \neq \phi$ and consider an ordered space E over the real scalars ordered by its positive cone with the assumption that $(E^+)^\oslash \neq \phi$. A function $p^E : \xi \times \xi \to E^+$ such that for all $\hbar, \vartheta, \varrho \in \xi$;*

(ρ_1) : $0_E \leq p^E(\hbar, \hbar) \leq p^E(\hbar, \vartheta)$.

(ρ_2) : $\hbar = \vartheta$ *if and only if* $p^E(\hbar, \hbar) = p^E(\hbar, \vartheta)$*'*

(ρ_3) : $p^E(\hbar, \vartheta) = p^E(\vartheta, \hbar)$,

(ρ_4) : $p^E(\hbar, \vartheta) \leq p^E(\hbar, \varrho) + p^E(\hbar, \vartheta) - p^E(\varrho, \varrho)$.

A pair (ξ, p^E) is known as partially E-cone metric space where $\xi \neq \phi$ and p^E is a partially E-cone metric on the set ξ.

Clearly, if $p^E(\hbar, \vartheta) = 0_E$, then from (ρ_1) and $(\rho_2), \hbar = \vartheta$. But if $\hbar = \vartheta, p^E(\hbar, \vartheta)$ may not be equal to 0_E.

Definition 2.3. [18] *Consider a partially E-cone metric (ξ, p^E) and let E be an ordered space with the assumption that $(E^+)^\oslash \neq \phi$. Consider a sequence $\{\hbar_n\}$ in ξ and $\hbar \in \xi$. Then*

(i) *If for every* $0_E \lll e$*, then, there exists a natural number* a *such that*

$$p^E(\hbar_n, \hbar) \lll e, \quad for\ all\ n \geq a.$$

Therefore, the sequence $\{\hbar_n\}$ *is called e-convergent to* $\hbar$.
Here, however, we write $\lim_{n\to\infty} \hbar_n = \hbar$ *or* $\hbar_n \xrightarrow{e} \hbar$.

(ii) *If for every* $0_E \lll e$*, then, there exists a natural number* a *such that*

$$p^E(\hbar_n, \hbar_m) \lll e, \quad for\ all\ n, m \geq a.$$

Therefore, the sequence $\{\hbar_n\}$ *is called e-Cauchy sequence.*

(iii) (ξ, p^E) *is e-complete if every e-Cauchy sequence is e-convergent.*

Definition 2.4. [12] *Consider a partially ordered set* $(\xi, \leq)$ *and* $\Im : \xi \times \xi \times \xi \to \xi$*. If for any* $\hbar, \vartheta, \varrho \in \xi$

$$\begin{aligned} \hbar_1, \hbar_2 \in \xi, &\quad \hbar_1 \leq \hbar_2 \Rightarrow \Im(\hbar_1, \vartheta, \varrho) \leq \Im(\hbar_2, \vartheta, \varrho),\\ \vartheta_1, \vartheta_2 \in \xi, &\quad \vartheta_1 \leq \vartheta_2 \Rightarrow \Im(\hbar, \vartheta_1, \varrho) \geq \Im(\hbar, \vartheta_2, \varrho),\\ \varrho_1, \varrho_2 \in \xi, &\quad \varrho_1 \leq \varrho_2 \Rightarrow \Im(\hbar, \vartheta, \varrho_1) \leq \Im(\hbar, \vartheta, \varrho_2). \end{aligned}$$

Then, the mapping $\Im$ *possesses the mixed monotone property.*

Definition 2.5. [12] *Let* $\Im : \xi \times \xi \times \xi \to \xi$ *be the mapping. An element* $(\hbar, \vartheta, \varrho)$, $\forall\ \hbar, \vartheta, \varrho \in \xi$ *is known as a tripled fixed point of* $\Im$ *if*

$$\Im(\hbar, \vartheta, \varrho) = \hbar, \quad \Im(\vartheta, \hbar, \vartheta) = \vartheta \ \ and \ \ \Im(\varrho, \vartheta, \hbar) = \varrho.$$

Berinde and Borcut additionally demonstrated the result specified below.

Theorem 2.1. [12] *Consider a partially ordered set* $(\xi, \leq, p^E)$ *and assume that there is an E-cone metric space* p^E *on* ξ *such that* (ξ, p^E) *is a complete E-cone metric space. Consider the mapping* $\Im : \xi \times \xi \times \xi \to \xi$ *such that* $\Im$ *possesses the mixed monotone property and there are* $i, j, k \geq 0$ *with* $i + j + k < 1$ *such that*

$$p^E\big(\Im(\hbar, \vartheta, \varrho), \Im(\hbar', \vartheta', \varrho')\big) \leq ip^E(\hbar, \hbar') + jp^E(\vartheta, \vartheta') + kp^E(\varrho, \varrho'), \tag{2.1}$$

for any $\hbar, \vartheta, \varrho \in \xi$ *for which* $\hbar \leq \hbar', \vartheta \geq \vartheta', \varrho \leq \varrho$*. Consider* $\Im$ *is continuous or* ξ *has the following property:*

(i) *if a non-decreasing sequence* $\hbar_n \to \hbar$*, then* $\hbar_n \leq \hbar\ \forall\ n$,

(ii) *if a non-increasing sequence* $\vartheta_n \to \vartheta$, *then* $\vartheta_n \geq \vartheta \ \forall \ n$.

If $\exists \ \hbar_0, \vartheta_0, \varrho_0 \in \xi$ *such that* $\hbar_0 \leq \Im(\hbar_0, \vartheta_0, \varrho_0), \vartheta_0 \geq \Im(\vartheta_0, \hbar_0, \vartheta_0)$ *and* $\varrho_0 \leq \Im(\varrho_0, \vartheta_0, \hbar_0)$, *then there exist* $\hbar, \vartheta, \varrho \in \xi$ *such that*

$$\Im(\hbar, \vartheta, \varrho) = \hbar, \quad \Im(\vartheta, \hbar, \vartheta) = \vartheta \quad and \quad \Im(\varrho, \vartheta, \hbar) = \varrho,$$

then, $\Im$ *has a triple fixed point.*

A few triple fixed point theorems for partially E-Cone metric space mappings with mixed monotone properties are presented in this study.

3. Main Results

Definition 3.1. *Consider a partial E-cone metric space* (ξ, p^E). *A mapping* $\wp : \xi \times \xi$ *is called ICS if* $\wp$ *is continuous, injective and has the property: for every sequence* $\{\hbar_n\}$ *in* ξ, *if* $\{\wp\hbar_n\}$ *is convergent then* $\{\hbar_n\}$ *is also convergent.*

Let φ *be the set of* $\psi : [0, +\infty) \to [0, +\infty)$ *such that*

(1) ψ *is non-decreasing,*

(2) $\psi(\imath) < \imath$ *for all* $\imath > 0$,

(3) $\lim_{\jmath \to \imath^+} \psi(\jmath) < \imath$ *for all* $\imath > 0$.

Theorem 3.1. *Consider a partially ordered set* $(\xi, \leq)$ *and assume that there is an E-cone metric* p^E *on* ξ *such that* (ξ, p^E) *is a complete E-cone metric space. Consider* $\wp : \xi \times \xi$ *is an ICS mapping and* $\Im : \xi \times \xi \times \xi \to \xi$ *is such that* $\Im$ *has the mixed monotone property. Suppose there exists* $\psi \in \varphi$ *such that*

$$p^E\big(\wp\Im(\hbar, \vartheta, \varrho), \wp\Im(\hbar', \vartheta', \varrho')\big) \leq \psi\Big(\max\left\{p^E(\wp\hbar, \wp\hbar'), p^E(\wp\vartheta, \wp\vartheta'), p^E(\wp\varrho, \wp\varrho')\right\}\Big) \quad (3.1)$$

for any $\hbar, \vartheta, \varrho \in \xi$ *for which* $\hbar \leq \hbar', \quad \vartheta \geq \vartheta' \quad and \quad \varrho \leq \varrho'$. *Suppose*

(i) $\Im$ *is continuous, or*

(ii) ξ *has the following property:*

(a) *if non-decreasing sequence* $\hbar_n \to \hbar$ (*correspondingly* $\varrho_n \to \varrho$), *then* $\hbar_n \leq \hbar$ (*correspondingly* $\varrho_n \leq \varrho$) *for all* n,

(b) *if non-increasing sequence* $\vartheta_n \to \vartheta$, *then* $\vartheta_n \geq \vartheta$ *for all* n.

If $\exists\ \hbar_0, \vartheta_0, \varrho_0 \in \xi$ *such that* $\hbar_0 \leq \Im(\hbar_0, \vartheta_0, \varrho_0), \vartheta_0 \geq \Im(\vartheta_0, \hbar_0, \vartheta_0)$ *and* $\varrho_0 \leq \Im(\varrho_0, \vartheta_0, \hbar_0)$, *then* $\exists\ \hbar, \vartheta, \varrho \in \xi$ *such that*

$$\Im(\hbar, \vartheta, \varrho) = \hbar, \quad \Im(\vartheta, \hbar, \vartheta) = \vartheta \ \ \textit{and} \ \ \Im(\varrho, \vartheta, \hbar) = \varrho,$$

then, $\Im$ *has a triple fixed point.*

Proof. Consider $\hbar_0, \vartheta_0, \varrho_0 \in \xi$ such that $\hbar_0 \leq \Im(\hbar_0, \vartheta_0, \varrho_0), \vartheta_0 \geq \Im(\vartheta_0, \hbar_0, \vartheta_0)$ and $\varrho_0 \leq \Im(\varrho_0, \vartheta_0, \hbar_0)$. Set

$$\hbar_1 = \Im(\hbar_0, \vartheta_0, \varrho_0), \quad \vartheta_1 = \Im(\vartheta_0, \hbar_0, \vartheta_0) \ \text{ and } \ \varrho_1 = \Im(\varrho_0, \vartheta_0, \hbar_0). \tag{3.2}$$

Continuing this process, we can construct sequences $\{\hbar_n\}, \{\vartheta_n\}$ and $\{\varrho_n\}$ in ξ such that

$$\hbar_{n+1} = \Im(\hbar_n, \vartheta_n, \varrho_n), \quad \vartheta_{n+1} = \Im(\vartheta_n, \hbar_n, \vartheta_n) \ \text{ and } \ \varrho_{n+1} = \Im(\varrho_n, \vartheta_n, \hbar_n). \tag{3.3}$$

Since $\Im$ possesses the mixed monotone property, by using a mathematical induction, we get

$$\hbar_n \leq \hbar_{n+1}, \quad \vartheta_n \geq \vartheta_{n+1}, \quad \varrho_n \leq \varrho_{n+1}, \text{ for } n = 0, 1, 2, ... \tag{3.4}$$

Suppose for $n \in \mathbb{N}$,

$$\hbar_n = \hbar_{n+1}, \quad \vartheta_n = \vartheta_{n+1}, \quad \text{and} \quad \varrho_n = \varrho_{n+1}$$

then, by (3.3), a triple fixed point of $\Im$ is $(\hbar_n, \vartheta_n, \varrho_n)$. For any $n \in \mathbb{N}$ such that

$$\hbar_n \neq \hbar_{n+1} \quad \textit{or} \quad \vartheta_n \neq \vartheta_{n+1} \quad \textit{or} \quad \varrho_n \neq \vartheta_{n+1}. \tag{3.5}$$

As $\wp$ is injective, then, for any $n \in \mathbb{N}$, by (3.5)

$$0 < \max\left\{p^E(\wp\hbar_n, \wp\hbar_{n+1}), p^E(\wp\vartheta_n, \wp\vartheta_{n+1}), p^E(\wp\varrho_n, \wp\varrho_{n+1})\right\}.$$

By (3.1) and (3.3), we obtain

$$\begin{aligned} p^E(\wp\hbar_n, \wp\hbar_{n+1}) &= p^E\big(\wp\Im(\hbar_{n-1}, \vartheta_{n-1}, \varrho_{n-1}), \wp\Im(\hbar_n, \vartheta_n, \varrho_n)\big) \\ &\leq \psi\Big(\max\left\{p^E(\wp\hbar_{n-1}, \wp\hbar_n), p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\varrho_{n-1}, \wp\varrho_n)\right\}\Big) \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
p^E(\wp\vartheta_{n+1}, \wp\vartheta_n) &= p^E\big(\wp\Im(\vartheta_n, \hbar_n, \vartheta_n), \wp\Im(\vartheta_{n-1}, \hbar_{n-1}, \vartheta_{n-1})\big) \\
&\leq \psi\Big(\big\{p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\hbar_{n-1}, \wp\hbar_n), p^E(\wp\vartheta_{n-1}, \wp\vartheta_n)\big\}\Big) \\
&= \psi\Big(\max\big\{p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\hbar_{n-1}, \wp\hbar_n)\big\}\Big) \\
&\leq \psi\Big(\max\big\{p^E(\wp\varrho_{n-1}, \wp\varrho_n), p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\hbar_{n-1}, \wp\hbar_n)\big\}\Big)
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
p^E(\wp\varrho_n, \wp\varrho_{n+1}) &= p^E\big(\wp\Im(\varrho_{n-1}, \vartheta_{n-1}, \hbar_{n-1}), \wp\Im(\varrho_n, \vartheta_n, \hbar_n)\big) \\
&\leq \psi\Big(\max\big\{p^E(\wp\varrho_{n-1}, \wp\varrho_n), p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\hbar_{n-1}, \wp\hbar_n)\big\}\Big).
\end{aligned} \tag{3.8}
$$

Since $\psi(\imath) < \imath$ for all $\imath > 0$, so from (3.6) to (3.8) we obtain that

$$
\begin{aligned}
0 &< \max\big\{p^E(\wp\hbar_n, \wp\hbar_{n+1}), p^E(\wp\vartheta_n, \wp\vartheta_{n+1}), p^E(\wp\varrho_n, \wp\varrho_{n+1})\big\} \\
&\leq \psi\Big(\max\big\{p^E(\wp\varrho_{n-1}, \wp\varrho_n), p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\hbar_{n-1}, \wp\hbar_n)\big\}\Big) \\
&< \max\big\{p^E(\wp\varrho_{n-1}, \wp\varrho_n), p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\hbar_{n-1}, \wp\hbar_n)\big\}.
\end{aligned} \tag{3.9}
$$

Then

$$
\begin{aligned}
&\max\big\{p^E(\wp\hbar_n, \wp\hbar_{n+1}), p^E(\wp\vartheta_n, \wp\vartheta_{n+1}), p^E(\wp\varrho_n, \wp\varrho_{n+1})\big\} \\
&\qquad < \max\big\{p^E(\wp\varrho_{n-1}, \wp\varrho_n), p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\hbar_{n-1}, \wp\hbar_n)\big\}.
\end{aligned}
$$

Therefore, $\Big\{\max\big\{p^E(\wp\hbar_n, \wp\hbar_{n+1}), p^E(\wp\vartheta_n, \wp\vartheta_{n+1}), p^E(\wp\varrho_n, \wp\varrho_{n+1})\big\}\Big\}$ is a positive decreasing sequence. Thus, $\exists\ a \geq 0$ such that

$$
\lim_{n\to+\infty} \max\big\{p^E(\wp\hbar_n, \wp\hbar_{n+1}), p^E(\wp\vartheta_n, \wp\vartheta_{n+1}), p^E(\wp\varrho_n, \wp\varrho_{n+1})\big\} = a.
$$

Consider $a > 0$. By applying $n \to +\infty$ in (3.9), we have

$$
\begin{aligned}
0 < a &\leq \lim_{n\to+\infty} \psi\Big(\max\big\{p^E(\wp\varrho_{n-1}, \wp\varrho_n), p^E(\wp\vartheta_{n-1}, \wp\vartheta_n), p^E(\wp\hbar_{n-1}, \wp\hbar_n)\big\}\Big) \\
&= \lim_{\imath\to\jmath^+} \psi(\imath) < a,
\end{aligned} \tag{3.10}
$$

which is a contradiction. Thus, we conclude that

$$
\lim_{n\to+\infty} \max\big\{p^E(\wp\hbar_n, \wp\hbar_{n+1}), p^E(\wp\vartheta_n, \wp\vartheta_{n+1}), p^E(\wp\varrho_n, \wp\varrho_{n+1})\big\} = 0 \tag{3.11}
$$

We claim that $\{\wp\hbar_n\}, \{\wp\vartheta_n\}$ and $\{\wp\varrho_n\}$ are Cauchy sequences. Suppose, to the contrary, that the sequence $\{\wp\hbar_n\}$, $\{\wp\vartheta_n\}$, or $\{\wp\varrho_n\}$ is not Cauchy. Then,

$$\lim_{n,m\to+\infty} p^E(\wp\hbar_m, \wp\hbar_n) \neq 0, \quad \text{or} \quad \lim_{n,m\to+\infty} p^E(\wp\vartheta_m, \wp\vartheta_n) \neq 0,$$

$$\text{or} \quad \lim_{n,m\to+\infty} p^E(\wp\varrho_m, \wp\varrho_n) \neq 0.$$

This implies that $\exists\ \varepsilon > 0$ for which we can determine the subsequences of integers (m_q) and (n_q) with $n_q > m_q > q$ such that

$$\max\left\{p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q}), p^E(\wp\vartheta_{m_q}, \wp\vartheta_{n_q}), p^E(\wp\varrho_{m_q}, \wp\varrho_{n_q})\right\} \geq \varepsilon. \tag{3.12}$$

Moreover, according to m_q we can choose the smallest integer n_q such that $n_q > m_q$ and satisfying (3.12). Then

$$\max\left\{p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q-1}), p^E(\wp\vartheta_{m_q}, \wp\vartheta_{n_q-1}), p^E(\wp\varrho_{m_q}, \wp\varrho_{n_q-1})\right\} < \varepsilon. \tag{3.13}$$

By triangle inequality and (3.13), we have

$$\begin{aligned} p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q}) &\leq p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q-1}) + p^E(\wp\hbar_{n_q-1}, \wp\hbar_{n_q}) \\ &< \varepsilon + p^E(\wp\hbar_{n_q-1}, \wp\hbar_{n_q}). \end{aligned}$$

Thus, by (3.11), we obtain

$$\lim_{q\to+\infty} p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q}) \leq \lim_{q\to+\infty} p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q-1}) \leq \varepsilon. \tag{3.14}$$

Similarly, we have

$$\lim_{q\to+\infty} p^E(\wp\vartheta_{m_q}, \wp\vartheta_{n_q}) \leq \lim_{q\to+\infty} p^E(\wp\vartheta_{m_q}, \wp\vartheta_{n_q-1}) \leq \varepsilon. \tag{3.15}$$

$$\lim_{q\to+\infty} p^E(\wp\varrho_{m_q}, \wp\varrho_{n_q}) \leq \lim_{q\to+\infty} p^E(\wp\varrho_{m_q}, \wp\varrho_{n_q-1}) \leq \varepsilon. \tag{3.16}$$

Again by (3.13)

$$\begin{aligned} p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q}) &\leq p^E(\wp\hbar_{m_q}, \wp\hbar_{m_q-1}) + p^E(\wp\hbar_{m_q-1}, \wp\hbar_{n_q-1}) + p^E(\wp\hbar_{n_q-1}, \wp\hbar_{n_q}) \\ &\leq p^E(\wp\hbar_{m_q}, \wp\hbar_{m_q-1}) + p^E(\wp\hbar_{m_q-1}, \wp\hbar_{m_q}) + p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q-1}) \\ &+ p^E(\wp\hbar_{n_q-1}, \wp\hbar_{n_q}) \\ &< p^E(\wp\hbar_{m_q}, \wp\hbar_{m_q-1}) + p^E(\wp\hbar_{m_q-1}, \wp\hbar_{m_q}) + \varepsilon + p^E(\wp\hbar_{n_q-1}, \wp\hbar_{n_q}). \end{aligned}$$

Taking $q \to +\infty$ and by (3.11), we obtain

$$\lim_{q\to+\infty} p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q}) \leq \lim_{q\to+\infty} p^E(\wp\hbar_{m_q-1}, \wp\hbar_{n_q-1}) \leq \varepsilon. \tag{3.17}$$

$$\lim_{q\to+\infty} p^E(\wp\vartheta_{m_q}, \wp\vartheta_{n_q}) \leq \lim_{q\to+\infty} p^E(\wp\vartheta_{m_q-1}, \wp\vartheta_{n_q-1}) \leq \varepsilon. \tag{3.18}$$

$$\lim_{q\to+\infty} p^E(\wp\varrho_{m_q}, \wp\varrho_{n_q}) \leq \lim_{q\to+\infty} p^E(\wp\varrho_{m_q-1}, \wp\varrho_{n_q-1}) \leq \varepsilon. \tag{3.19}$$

Using (3.12) and (3.17 to 3.19), we have

$$\begin{aligned} &\lim_{q\to+\infty} \max\left\{p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q}), p^E(\wp\vartheta_{m_q}, \wp\vartheta_{n_q}), p^E(\wp\varrho_{m_q}, \wp\varrho_{n_q})\right\} \\ &= \lim_{q\to+\infty} \max\left\{p^E(\wp\hbar_{m_q-1}, \wp\hbar_{n_q-1}), p^E(\wp\vartheta_{m_q-1}, \wp\vartheta_{n_q-1}), p^E(\wp\varrho_{m_q-1}, \wp\varrho_{n_q-1})\right\} \\ &= \varepsilon. \end{aligned} \tag{3.20}$$

By using the inequality (3.1), we have

$$\begin{aligned} &p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q}) = p^E\big(\wp\Im(\hbar_{m_q-1}, \vartheta_{m_q-1}, \varrho_{m_q-1})\wp\Im(\hbar_{n_q-1}, \vartheta_{n_q-1}, \varrho_{n_q-1})\big) \\ &\leq \psi\Big(\max\left\{p^E(\wp\hbar_{m_q-1}, \wp\hbar_{n_q-1}), p^E(\wp\vartheta_{m_q-1}, \wp\vartheta_{n_q-1}), p^E(\wp\varrho_{m_q-1}, \wp\varrho_{n_q-1})\right\}\Big) \end{aligned} \tag{3.21}$$

$$\begin{aligned} p^E(\wp\vartheta_{m_q}, \wp\vartheta_{n_q}) &= p^E\big(\wp\Im(\vartheta_{m_q-1}, \hbar_{m_q-1}, \vartheta_{m_q-1}), \wp\Im(\vartheta_{n_q-1}, \hbar_{n_q-1}, \varrho_{n_q-1})\big) \\ &\leq \psi\Big(\max\left\{p^E(\wp\vartheta_{m_q-1}, \wp\vartheta_{n_q-1}), p^E(\wp\hbar_{m_q-1}, \wp\hbar_{n_q-1})\right\}\Big) \end{aligned} \tag{3.22}$$

And

$$\begin{aligned} &p^E(\wp\varrho_{m_q}, \wp\varrho_{n_q}) = p^E\big(\wp\Im(\varrho_{m_q-1}, \vartheta_{m_q-1}, \hbar_{m_q-1}), \wp\Im(\varrho_{n_q-1}, \vartheta_{n_q-1}, \hbar_{n_q-1})\big) \\ &\leq \psi\Big(\max\left\{p^E(\wp\hbar_{m_q-1}, \wp\hbar_{n_q-1}), p^E(\wp\vartheta_{m_q-1}, \wp\vartheta_{n_q-1}), p^E(\wp\varrho_{m_q-1}, \wp\varrho_{n_q-1})\right\}\Big) \end{aligned} \tag{3.23}$$

From (3.21, 3.22 and 3.23), we deduce that

$$\begin{aligned} &\max\left\{p^E(\wp\hbar_{m_q}, \wp\hbar_{n_q}), p^E(\wp\vartheta_{m_q}, \wp\vartheta_{n_q}), p^E(\wp\varrho_{m_q}, \wp\varrho_{n_q})\right\} \\ &\leq \psi\Big(\max\left\{p^E(\wp\hbar_{m_q-1}, \wp\hbar_{n_q-1}), p^E(\wp\vartheta_{m_q-1}, \wp\vartheta_{n_q-1}), p^E(\wp\varrho_{m_q-1}, \wp\varrho_{n_q-1})\right\}\Big) \end{aligned} \tag{3.24}$$

Taking $q \to +\infty$ in (3.24) and from (3.20), we get

$$0 < \varepsilon \leq \lim_{\imath\to\varepsilon^+} \psi(\imath) < \varepsilon,$$

which is a contradiction. Hence, $\{\wp\hbar_n\}, \{\wp\vartheta_n\}$ and $\{\wp\varrho_n\}$ are Cauchy sequences in (ξ, p^E). $\{\wp\hbar_n\}, \{\wp\vartheta_n\}$ and $\{\wp\varrho_n\}$ are convergent sequences, as ξ is a complete E-cone metric space.

Since $\wp$ is an ICS mapping, $\exists\ \hbar, \vartheta, \varrho \in \xi$ such that

$$\lim_{n\to+\infty} \hbar_n = \hbar, \quad \lim_{n\to+\infty} \vartheta_n = \vartheta, \quad \text{and} \quad \lim_{n\to+\infty} \varrho_n = \varrho. \tag{3.25}$$

Since T is continuous, we have

$$\lim_{n\to+\infty} \wp\hbar_n = \wp\hbar, \quad \lim_{n\to+\infty} \wp\vartheta_n = \wp\vartheta, \quad \text{and} \quad \lim_{n\to+\infty} \wp\varrho_n = \wp\varrho. \tag{3.26}$$

Now, consider (i) holds, that is $\Im$ is continuous. Then from (3.3), (3.25) and (3.26), we get

$$\hbar = \lim_{n\to+\infty} \hbar_{n+1} = \lim_{n\to+\infty} \Im(\hbar_n, \vartheta_n, \varrho_n) = \Im(\lim_{n\to+\infty} \hbar_n, \lim_{n\to+\infty} \vartheta_n, \lim_{n\to+\infty} \varrho_n) = \Im(\hbar, \vartheta, \varrho),$$

$$\vartheta = \lim_{n\to+\infty} \vartheta_{n+1} = \lim_{n\to+\infty} \Im(\vartheta_n, \hbar_n, \vartheta_n) = \Im(\lim_{n\to+\infty} \vartheta_n, \lim_{n\to+\infty} \hbar_n, \lim_{n\to+\infty} \vartheta_n) = \Im(\vartheta, \hbar, \vartheta),$$

and

$$\varrho = \lim_{n\to+\infty} \varrho_{n+1} = \lim_{n\to+\infty} \Im(\varrho_n, \vartheta_n, \hbar_n) = \Im(\lim_{n\to+\infty} \varrho_n, \lim_{n\to+\infty} \vartheta_n, \lim_{n\to+\infty} \hbar_n) = \Im(\varrho, \vartheta, \hbar),$$

to proved that $\Im$ has a triple fixed point.

Consider (ii) holds. Since $\{\hbar_n\}, \{\varrho_n\}$ are non-decreasing with $\hbar_n \to \hbar, \varrho_n \to \varrho$ and $\{\vartheta_n\}$ is also non-increasing with $\vartheta_n \to \vartheta$, then by (ii) we obtain

$$\hbar_n \leq \hbar, \quad \vartheta_n \geq \vartheta \quad \text{and} \quad \varrho_n \leq \varrho, \quad \text{for all } n.$$

Consider now

$$\begin{aligned} p^E\big(\wp\hbar, \wp\Im(\hbar, \vartheta, \varrho)\big) &\leq p^E(\wp\hbar, \wp\hbar_{n+1}) + p^E\big(\wp\hbar_{n+1}, \wp\Im(\hbar, \vartheta, \varrho)\big) \\ &= p^E(\wp\hbar, \wp\hbar_{n+1}) + p^E\big(\wp\Im(\hbar_n, \vartheta_n, \varrho_n), \wp\Im(\hbar, \vartheta, \varrho)\big) \\ &\leq p^E(\wp\hbar, \wp\hbar_{n+1}) + \psi\Big(\max\big\{p^E(\wp\hbar_n, \wp\hbar), p^E(\wp\vartheta_n, \wp\vartheta), p^E(\wp\varrho_n, \wp\varrho)\big\}\Big). \end{aligned} \tag{3.27}$$

Letting $n \to +\infty$ and by using (3.26), the right-hand side of (3.27) tends to 0, thus we obtain $p^E\big(\wp\hbar, \wp\Im(\hbar, \vartheta, \varrho)\big) = 0$. Therefore, $\wp\hbar = \wp\Im(\hbar, \vartheta, \varrho)$ and since $\wp$ is injective, we have $\hbar = \Im(\hbar, \vartheta, \varrho)$. Analogously, we find that

$$\Im(\vartheta, \hbar, \vartheta) = \vartheta \quad \text{and} \quad \Im(\varrho, \vartheta, \hbar) = \varrho.$$

Thus, $\Im$ has a triple fixed point.

Corollary 3.1. *Consider a partially ordered set $(\xi, \leq)$ and assume that there is an E-cone metric p^E on ξ such that (ξ, p^E) is a complete E-cone metric space. Consider $\wp : \xi \to \xi$ is an ICS mapping and $\Im : \xi \times \xi \times \xi \to \xi$ is such that $\Im$ has the mixed monotone property. Suppose $\exists\ \psi \in \varphi$ such that*

$$p^E\big(\wp\Im(\hbar,\vartheta,\varrho),\wp\Im(\hbar',\vartheta',\varrho')\big)\leq\psi\left(\frac{p^E(\wp\hbar,\wp\hbar')+p^E(\wp\vartheta,\wp\vartheta')+p^E(\wp\varrho,\wp\varrho')}{3}\right)$$

for any $\hbar,\vartheta,\varrho\in\xi$ *for which* $\hbar\leq\hbar'$, $\vartheta\geq\vartheta'$ *and* $\varrho\leq\varrho'$. *Suppose*

(i) $\Im$ *is continuous, or*

(ii) ξ *has the following property:*

- *(a) if non-decreasing sequence* $\hbar_n\to\hbar$ *(correspondingly* $\varrho_n\to\varrho$*), then* $\hbar_n\leq\hbar$ *(correspondingly* $\varrho_n\leq\varrho$*) for all* n,
- *(b) if non-increasing sequence* $\vartheta_n\to\vartheta$, *then* $\vartheta_n\geq\vartheta$ *for all* n.

If $\exists\ \hbar_0,\vartheta_0,\varrho_0\in\xi$ *such that* $\hbar_0\leq\Im(\hbar_0,\vartheta_0,\varrho_0)$, $\vartheta_0\geq\Im(\vartheta_0,\hbar_0,\vartheta_0)$ *and* $\varrho_0\leq\Im(\varrho_),\vartheta_0,\hbar_0)$, *then* $\exists\ \hbar,\vartheta,\varrho\in\xi$ *such that*

$$\Im(\hbar,\vartheta,\varrho)=\hbar,\quad\Im(\vartheta,\hbar,\vartheta)=\vartheta\quad\text{and}\quad\Im(\varrho,\vartheta,\hbar)=\varrho,$$

then, $\Im$ *has a triple fixed point.*

Proof. It is sufficient to say that

$$\frac{p^E(\wp\hbar,\wp\hbar')+p^E(\wp\vartheta,\wp\vartheta')+p^E(\wp\varrho,\wp\varrho')}{3}\leq\max\big\{p^E(\wp\hbar,\wp\hbar'),p^E(\wp\vartheta,\wp\vartheta'),p^E(\wp\varrho,\wp\varrho')\big\}.$$

Then, applying Theorem 3.1 so that ψ is non-decreasing.

Corollary 3.2. *Consider a partially ordered set* $(\xi,\leq)$ *and assume that there is an E-cone metric* p^E *on* ξ *such that* (ξ,p^E) *is a complete E-cone metric space. Consider* $\wp:\xi\to\xi$ *is an ICS mapping and* $\Im:\xi\times\xi\times\xi\to\xi$ *is such that* $\Im$ *has the mixed monotone property. Suppose* $\exists\ q\in[0,1)$ *such that*

$$p^E\big(\wp\Im(\hbar,\vartheta,\varrho),\wp\Im(\hbar',\vartheta',\varrho')\big)\leq q\max\big\{p^E(\wp\hbar,\wp\hbar'),p^E(\wp\vartheta,\wp\vartheta'),p^E(\wp\varrho,\wp\varrho')\big\}$$

for any $\hbar,\vartheta,\varrho\in\xi$ *for which* $\hbar\leq\hbar'$, $\vartheta\geq\vartheta'$ *and* $\varrho\leq\varrho'$. *Suppose*

(i) $\Im$ *is continuous, or*

(ii) ξ *has the following property:*

- *(a) if non-decreasing sequence* $\hbar_n\to\hbar$ *(Correspondingly* $\varrho_n\to\varrho$*), then* $\hbar_n\leq\hbar$ *(Correspondingly* $\varrho_n\leq\varrho$*) for all* n,
- *(b) if non-increasing sequence* $\vartheta_n\to\vartheta$, *then* $\vartheta_n\geq\vartheta$ *for all* n.

If $\exists\ \hbar_0, \vartheta_0, \varrho_0 \in \xi$ *such that* $\hbar_0 \leq \Im(\hbar_0, \vartheta_0, \varrho_0)$, $\vartheta_0 \geq \Im(\vartheta_0, \hbar_0, \vartheta_0)$ *and* $\varrho_0 \leq \Im(\varrho_0, \vartheta_0, \hbar_0)$, *then* $\exists\ \hbar, \vartheta, \varrho \in \xi$ *such that*

$$\Im(\hbar, \vartheta, \varrho) = \hbar, \quad \Im(\vartheta, \hbar, \vartheta) = \vartheta \quad \textit{and} \quad \Im(\varrho, \vartheta, \hbar) = \varrho,$$

then, $\Im$ *has a triple fixed point.*

Proof. The proof is complete by taking $\psi(\imath) = q\imath$ in Theorem 3.1.

Corollary 3.3. *Consider a partially ordered set* $(\xi, \leq)$ *and assume that there is an E-cone metric* p^E *on* ξ *such that* (ξ, p^E) *is a complete E-cone metric space. Consider* $\wp : \xi \to \xi$ *is an ICS mapping and* $\Im : \xi \times \xi \times \xi \to \xi$ *is such that* $\Im$ *has the mixed monotone property. Suppose* $\exists\ q \in [0, 1)$ *such that*

$$p^E\big(\wp\Im(\hbar, \vartheta, \varrho), \wp\Im(\hbar', \vartheta', \varrho') \leq \frac{q}{3}\big(p^E(\wp\hbar, \wp\hbar') + p^E(\wp\vartheta, \wp\vartheta') + p^E(\wp\varrho, \wp\varrho')\big) \tag{3.28}$$

for any $\hbar, \vartheta, \varrho, \hbar', \vartheta', \varrho' \in \xi$ *for which* $\hbar \leq \hbar'$, $\vartheta \geq \vartheta'$ *and* $\varrho \leq \varrho'$. *Suppose*

(i) $\Im$ *is continuous, or*

(ii) ξ *has the following property:*

- *(a) if non-decreasing sequence* $\hbar_n \to \hbar$ *(correspondingly,* $\varrho_n \to \varrho$*), then* $\hbar_n \leq \hbar$ *(correspondingly,* $\varrho_n \leq \varrho$*) for all n,*
- *(b) if non-increasing sequence* $\vartheta_n \to \vartheta$*, then* $\vartheta_n \geq \vartheta$ *for all n.*

If $\exists\ \hbar_0, \vartheta_0, \varrho_0 \in \xi$ *such that* $\hbar_0 \leq \Im(\hbar_0, \vartheta_0, \varrho_0)$, $\vartheta_0 \geq \Im(\vartheta_0, \hbar_0, \vartheta_0)$ *and* $\varrho_0 \leq \Im(\varrho_0, \vartheta_0, \hbar_0)$, *then* $\exists\ \hbar, \vartheta, \varrho \in \xi$ *such that*

$$\Im(\hbar, \vartheta, \varrho) = \hbar, \quad \Im(\vartheta, \hbar, \vartheta) = \vartheta \quad \textit{and} \quad \Im(\varrho, \vartheta, \hbar) = \varrho,$$

then, $\Im$ *has a triple fixed point.*

Proof. The proof is complete by taking $\psi(\imath) = q\imath$ in Theorem 3.1.

Remark 3.1. *Consider the identity* $\wp = Ip^E_\xi$ *on* ξ*, in Corollary 3.3, we obtain Berinde and Borcut's Theorem (Theorem2.1), with* $i = j = k = \frac{q}{3}$.

To prove that the existence of triple fixed point and it is unique. Consider a partially ordered set $(\xi, \leq)$ and for $\xi \times \xi \times \xi$, define a partial ordering as: $\forall\ (\hbar, \vartheta, \varrho), (\hbar', \vartheta', \varrho') \in \xi \times \xi \times \xi$

$$(\hbar, \vartheta, \varrho) \leq (\hbar', \vartheta', \varrho') \Leftrightarrow \hbar \leq \hbar', \quad \vartheta \geq \vartheta' \quad \text{and} \quad \varrho \leq \varrho'. \tag{3.29}$$

We say that $(\hbar, \vartheta, \varrho)$ and $(\hbar', \vartheta', \varrho')$ are comparable if

$$(\hbar,\vartheta,\varrho) \leq (\hbar',\vartheta',\varrho') \quad \text{or} \quad (\hbar',\vartheta',\varrho') \leq (\hbar,\vartheta,\varrho).$$

Thus, $(\hbar,\vartheta,\varrho)$ is equal to $(\hbar',\vartheta',\varrho')$ if and only if $\hbar=\hbar'$, $\vartheta=\vartheta'$ and $\varrho=\varrho'$.

Theorem 3.2. *In addition to hypothesis of Theorem 3.1, Assume that for all* $(\hbar,\vartheta,\varrho),(\hbar',\vartheta',\varrho') \in \xi\times\xi\times\xi$, *there exist* $(u,v,w)\in\xi\times\xi\times\xi$ *such that* $\big(\Im(u,v,w), \Im(v,u,v),\Im(w,v,u)\big)$ *is comparable to* $\big(\Im(\hbar,\vartheta,\varrho),\Im(\vartheta,\hbar,\vartheta),\Im(\varrho,\vartheta,\hbar)\big)$ *and* $\big(\Im(\hbar',\vartheta',\varrho'),\Im(\vartheta',\hbar',\vartheta'),\Im(\varrho',\vartheta',\hbar')\big)$. *Then* $\Im$ *has a unique triple fixed point* $(\hbar,\vartheta,\varrho)$.

Proof. By Theorem 3.1, the set of triple fixed points of $\Im$ is not empty. Suppose, $(\hbar,\vartheta,\varrho)$ and $(\hbar',\vartheta',\varrho')$ are two triple fixed points of $\Im$, that is,

$$\Im(\hbar,\vartheta,\varrho)=\hbar, \quad \Im(\hbar',\vartheta',\varrho')=\hbar',$$
$$\Im(\vartheta,\hbar,\vartheta)=\vartheta, \quad \Im(\vartheta',\hbar',\vartheta')=\vartheta',$$
$$\Im(\varrho,\vartheta,\hbar)=\varrho, \quad \Im(\varrho',\vartheta',\hbar')=\varrho'.$$

Now we have to show that $(\hbar,\vartheta,\varrho)$ and $(\hbar',\vartheta',\varrho')$ are equal. By assumption, $\exists\ (u,v,w)\in\xi\times\xi\times\xi$ such that $\big(\Im(u,v,w),\Im(v,u,v),\Im(w,v,u)\big)$ is comparable to

$$\big(\Im(\hbar,\vartheta,\varrho),\Im(\vartheta,\hbar,\vartheta),\Im(\varrho,\vartheta,\hbar)\big)$$

and

$$\big(\Im(\hbar',\vartheta',\varrho'),\Im(\vartheta',\hbar',\vartheta'),\Im(\varrho',\vartheta',\hbar')\big).$$

Define sequences $\{u_n\},\{v_n\}$ and $\{w_n\}$ such that

$$u_0=u, \quad v_0=v, \quad w_0=w, \quad \text{and for any} \quad n\geq 1$$

$$\begin{aligned} u_n &= \Im(u_{n-1},v_{n-1},w_{n-1}),\\ v_n &= \Im(v_{n-1},u_{n-1},v_{n-1}),\\ w_n &= \Im(w_{n-1},v_{n-1},u_{n-1}), \quad \forall\ n. \end{aligned} \tag{3.30}$$

Moreover, set $\hbar_0=\hbar,\vartheta_0=\vartheta,\varrho_0=\varrho$ and $\hbar'_0=\hbar',\vartheta'_0=\vartheta',\varrho'_0=\varrho'$, and similarly define the sequences $\{\hbar_n\},\{\vartheta_n\},\{\varrho_n\}$ and $\{\hbar'_n\},\{\vartheta'_n\},\{\varrho'_n\}$. Then,

$$\begin{aligned} \hbar_n &= \Im(\hbar,\vartheta,\varrho), \quad \hbar'_n = \Im(\hbar',\vartheta',\varrho'),\\ \vartheta_n &= \Im(\vartheta,\hbar,\vartheta), \quad \vartheta'_n = \Im(\vartheta',\hbar',\vartheta'),\\ \varrho_n &= \Im(\varrho,\vartheta,\hbar), \quad \varrho'_n = \Im(\varrho',\vartheta',\hbar'), \quad \text{for all } n\geq 1. \end{aligned} \tag{3.31}$$

Since $\big(\Im(\hbar,\vartheta,\varrho),\Im(\vartheta,\hbar,\vartheta),\Im(\varrho,\vartheta,\hbar)\big)=(\hbar_1,\vartheta_1,\varrho_1,)=(\hbar,\vartheta,\varrho)$ is comparable to

$$\big(\Im(u,v,w),\Im(v,u,v),\Im(w,v,u)\big)=(u_1,v_1,w_1),$$

then, it is easy to show $(\hbar,\vartheta,\varrho)\geq(u_1,v_1,w_1)$. Recursively, we get that

$$(\hbar, \vartheta, \varrho) \geq (u_n, v_n, w_n) \quad \text{for all } n. \tag{3.32}$$

By (3.32) and (3.1), we have

$$\begin{aligned} p^E(\wp\hbar, \wp u_{n+1}) &= p^E\big(\wp\Im(\hbar, \vartheta, \varrho), \wp\Im(u_n, v_n, w_n)\big) \\ &\leq \psi\Big(\max\big\{p^E(\wp\hbar, \wp u_n), p^E(\wp\vartheta, \wp v_n), p^E(\wp\varrho, \wp w_n)\big\}\Big) \end{aligned} \tag{3.33}$$

$$\begin{aligned} p^E(\wp v_{n+1}, \wp\vartheta) &= p^E\big(\wp\Im(v_n, u_n, v_n), \wp\Im(\vartheta, \hbar, \vartheta)\big) \\ &\leq \psi\Big(\max\big\{p^E(\wp u_n, \wp\hbar), p^E(\wp v_n, \wp\vartheta)\big\}\Big) \\ &\leq \psi\Big(\max\big\{p^E(\wp v_n, \wp\vartheta), p^E(\wp u_n, \wp\hbar), p^E(\wp w_n, \wp\varrho)\big\}\Big) \end{aligned} \tag{3.34}$$

$$\begin{aligned} p^E(\wp\varrho, \wp w_{n+1}) &= p^E\big(\wp\Im(\varrho, \vartheta, \hbar), \wp\Im(w_n, v_n, u_n)\big) \\ &\leq \psi\Big(\max\big\{p^E(\wp\varrho, \wp w_n), p^E(\wp\vartheta, \wp v_n), p^E(\wp\hbar, \wp u_n)\big\}\Big) \end{aligned} \tag{3.35}$$

It follows from (3.33) to (3.35) that

$$\begin{aligned} \max\big\{p^E(\wp\varrho, \wp w_{n+1}),&p^E(\wp\vartheta, v_{n+1}), p^E(\wp\hbar, \wp u_{n+1})\big\} \\ &\leq \psi\Big(\max\big\{p^E(\wp\varrho, \wp w_n), p^E(\wp\vartheta, \wp v_n), p^E(\wp\hbar, \wp u_n)\big\}\Big). \end{aligned}$$

Thus, for each $n \geq 1$,

$$\begin{aligned} \max\big\{p^E(\wp\varrho, \wp w_n),&p^E(\wp\vartheta, \wp v_n), p^E(\wp\hbar, \wp u_n)\big\} \\ &\leq \psi^n\Big(\max\big\{p^E(\wp\varrho, \wp w_0), p^E(\wp\vartheta, \wp v_0), p^E(\wp\hbar, \wp u_0)\big\}\Big). \end{aligned} \tag{3.36}$$

Since $\psi(\imath) < \imath$ and $\lim\limits_{\jmath \to \imath^+} \psi(\jmath) < \imath$ imply $\lim\limits_{n \to +\infty} \psi^n(\imath) = 0$ for each $\imath > 0$.

Therefore, from (3.36),

$$\lim_{n \to +\infty} \max\big\{p^E(\wp\varrho, \wp w_n), p^E(\wp\vartheta, \wp v_n), p^E(\wp\hbar, \wp u_n)\big\} = 0.$$

Consequently, that

$$\lim_{n \to +\infty} p^E(\wp\hbar, \wp u_n) = 0, \quad \lim_{n \to +\infty} p^E(\wp\vartheta, \wp v_n) = 0, \quad \lim_{n \to +\infty} p^E(\wp\varrho, \wp w_n) = 0. \tag{3.37}$$

Similarly, we show that

$$\lim_{n \to +\infty} p^E(\wp\hbar', \wp u_n) = 0, \quad \lim_{n \to +\infty} p^E(\wp\vartheta', \wp v_n) = 0, \quad \lim_{n \to +\infty} p^E(\wp\varrho', \wp w_n) = 0. \tag{3.38}$$

Combining (3.37) and (3.28) gives that $(\wp\hbar, \wp\vartheta, \wp\varrho)$ and $(\wp\hbar', \wp\vartheta', \wp\varrho')$ are equal. Since $\wp$ is injective, we have $\hbar = \hbar', \vartheta = \vartheta'$ and $\varrho = \varrho'$.

4. Conclusion

In conclusion, this study has successfully extended the fixed-point theory to tripled fixed points in complete E-cone metric spaces, offering a broader framework for solving nonlinear problems. By establishing new contractive conditions, we demonstrated the existence and uniqueness of tripled fixed points under specific mappings. These results significantly generalize previous findings in traditional metric and cone metric spaces, providing a versatile tool for analyzing complex systems in various fields, including dynamic systems, optimization, and differential equations. The insights gained through this research open new pathways for further exploration in the study of fixed points in generalized spaces and their potential applications in mathematical modeling and applied sciences.

Acknowledgement

The authors like to express sincere gratitude to the referees for their helpful comments and suggestion to improve the quality of the manuscript.

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